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PROBLEM SET #2

1/22/15

1. free nonlinear oscillator satisfying:
$$\ddot{x} + w_0^2 x = -\alpha x^2 - \beta x^3$$

*assuming the nonlinear terms are due to an expansion in some small parameter, take $\alpha = O(\epsilon)$, $\beta = O(\epsilon^2)$

let $\underset{\sim\epsilon^0}{x} = \underset{\sim\epsilon^0}{x^{(1)}} + \underset{\sim\epsilon^1}{x^{(2)}} + \underset{\sim\epsilon^2}{x^{(3)}} + \ldots$, $w = \underset{\sim\epsilon^0}{w_0} + \underset{\sim\epsilon^1}{w^{(1)}} + \underset{\sim\epsilon^2}{w^{(2)}} + \ldots$

$x^{(1)} = a\cos wt \longrightarrow$ solution when $\alpha, \beta \to 0 \Rightarrow w \to w_0$

Rewrite equation of motion with extra terms:

$$\frac{w_0^2}{w^2}\ddot{x} + w_0^2 x = -\alpha x^2 - \beta x^3 - \left(1 - \frac{w_0^2}{w^2}\right)\ddot{x}$$

$$\frac{w_0^2}{w^2}\left(\cancel{-w^2 x^{(1)}} + \ddot{x}^{(2)}\right) + w_0^2\left(\cancel{x^{(1)}} + x^{(2)}\right)$$
$$= -\alpha\left(x^{(1)} + x^{(2)}\right)^2 - \beta\left(x^{(1)} + x^{(2)}\right)^3 - \left(1 - \frac{w_0^2}{w^2}\right)\left(-w^2 x^{(1)} + \ddot{x}^{(2)}\right)$$

eliminating higher order terms ($O(\epsilon^2)$)
$$\frac{w_0^2}{w^2}\ddot{x}^{(2)} + w_0^2 x^{(2)} = -\alpha\left(x^{(1)2} + \cancel{2x^{(1)}x^{(2)}} + \cancel{x^{(2)2}}\right) - \beta\left(\cancel{x^{(1)3}} + \cancel{3x^{(1)2}x^{(2)}} + \cancel{3x^{(1)}x^{(2)2}} + \cancel{x^{(2)3}}\right)$$
$$+ \left(w^2 - w_0^2\right)x^{(1)} - \left(1 - \frac{w_0^2}{w^2}\right)\ddot{x}^{(2)}$$

$$\ddot{x}^{(2)} + w_0^2 x^{(2)} = -\alpha x^{(1)2} + \left[\left(w_0 + w^{(1)}\right)^2 - w_0^2\right]x^{(1)}$$
$$= -\alpha x^{(1)2} + \left(w_0^2 + 2w_0 w^{(1)} + \cancel{w^{(1)2}} - w_0^2\right)x^{(1)} \qquad w^{(1)2} = O(\epsilon^2)$$
$$= -\alpha x^{(1)2} + 2w_0 w^{(1)} x^{(1)}$$
$$= -\alpha a^2\cos^2 wt + 2w_0 w^{(1)} a\cos wt$$
$$= -\frac{1}{2}\alpha a^2 - \frac{1}{2}\alpha a^2\cos(2wt) + \underbrace{2w_0 w^{(1)} a\cos wt}$$

$\Rightarrow$ need $\boxed{w^{(1)} = 0}$ so that this resonant term goes to 0

$$\Rightarrow \ddot{x}^{(2)} + w_0^2 x^{(2)} = -\frac{1}{2}\alpha a^2 - \frac{1}{2}\alpha a^2\cos(2wt)$$

let $x^{(2)} = A + B\cos(2wt)$ $\qquad w = w_0 + w^{(1)} = w_0$
$$\dot{x}^{(2)} = 0 - 2wB\sin(2wt)$$
$$\ddot{x}^{(2)} = -4w^2 B\cos(2wt) = -4w_0^2 B\cos(2wt)$$

$$-4w_0^2 B\cos(2wt) + Aw_0^2 + w_0^2 B\cos(2wt) = -\frac{1}{2}\alpha a^2 - \frac{1}{2}\alpha a^2\cos(2wt)$$

$$A = -\frac{1}{2}\frac{\alpha a^2}{w_0^2} \quad B = \frac{1}{6}\frac{\alpha a^2}{w_0^2} \Rightarrow \boxed{x^{(2)} = -\frac{\alpha a^2}{2w_0^2} + \frac{\alpha a^2}{6w_0^2}\cos(2wt)} \longrightarrow$$

1. (cont'd)

$$\frac{\omega_0^2}{\omega^2}\ddot{x} + \omega_0^2 x = -\alpha x^2 - \beta x^3 - \left(1-\frac{\omega_0^2}{\omega^2}\right)\ddot{x}$$

→ substitute $x = x^{(1)} + x^{(2)} + x^{(3)}$, $\omega = \omega_0 + \omega^{(1)} + \omega^{(2)} = \omega_0 + \omega^{(2)}$

$$\frac{\omega_0^2}{\omega^2}\left(\ddot{x}^{(1)} + \ddot{x}^{(2)} + \ddot{x}^{(3)}\right) + \omega_0^2\left(x^{(1)} + x^{(2)} + x^{(3)}\right) = -\alpha\left(x^{(1)} + x^{(2)} + x^{(3)}\right)^2 - \beta\left(x^{(1)} + x^{(2)} + x^{(3)}\right)^3$$
$$-\left(1-\frac{\omega_0^2}{\omega^2}\right)\left(\ddot{x}^{(1)} + \ddot{x}^{(2)} + \ddot{x}^{(3)}\right)$$

LHS: $$\frac{\omega_0^2}{\omega^2}\left(\cancel{-\omega^2 a\cos\omega t} - \frac{2}{3}\frac{\omega^2}{\omega_0^2}\alpha a^2\cos(2\omega t) + \ddot{x}^{(3)}\right) + \omega_0^2\left(\cancel{a\cos\omega t} - \frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{6\omega_0^2}\cos(2\omega t) + x^{(3)}\right)$$

$$= -\frac{2}{3}\alpha a^2\cos(2\omega t) + \frac{\omega_0^2}{\omega^2}\ddot{x}^{(3)} - \frac{\alpha a^2}{2} + \frac{\alpha a^2}{6}\cos(2\omega t) + \omega_0^2 x^{(3)}$$

RHS: $$-\alpha\left(x^{(1)2} + \cancel{x^{(2)2} + x^{(3)2}} + 2x^{(1)}x^{(2)} + \cancel{2x^{(1)}x^{(3)} + 2x^{(2)}x^{(3)}}\right)$$
$$-\beta\left(x^{(1)3} + \cancel{x^{(2)3} + x^{(3)3} + 3x^{(1)2}x^{(2)} + 3x^{(1)2}x^{(3)} + \ldots \text{(higher order terms)}}\right)$$
$$-\left(1-\frac{\omega_0^2}{\omega^2}\right)\left(\ddot{x}^{(1)} + \ddot{x}^{(2)} + \ddot{x}^{(3)}\right)$$

eliminate terms $O(\epsilon^3)$ or higher

$$\approx -\alpha x^{(1)2} - 2\alpha x^{(1)}x^{(2)} - \beta x^{(1)3} + (\omega^2-\omega_0^2)x^{(1)} - \left(1-\frac{\omega_0^2}{\omega^2}\right)\ddot{x}^{(2)} - \left(1-\frac{\omega_0^2}{\omega^2}\right)\ddot{x}^{(3)}$$

$$x^{(1)}x^{(2)} = -\frac{\alpha a^3}{2\omega_0^2}\cos\omega t + \frac{\alpha a^3}{6\omega_0^2}\cos\omega t\cos 2\omega t$$

$$(\omega^2-\omega_0^2) = \left[(\omega_0+\omega^{(2)})^2 - \omega_0^2\right] = 2\omega_0\omega^{(2)} + \omega^{(2)2}$$

$$\ddot{x}^{(2)} = -\frac{2}{3}\alpha a^2\left(\frac{\omega^2}{\omega_0^2}\right)\cos(2\omega t)$$

LHS=RHS: $$\cancel{\frac{\omega_0^2}{\omega^2}}\ddot{x}^{(3)} + \omega_0^2 x^{(3)} - \frac{\alpha a^2}{2}\cos(2\omega t) - \frac{\alpha a^2}{2} = -\alpha a^2\cos^2\omega t + \frac{\alpha^2 a^3}{\omega_0^2}\cos\omega t - \frac{\alpha^2 a^3}{3\omega_0^2}\cos\omega t\cos(2\omega t)$$
$$-\beta a^3\cos^3\omega t + 2\omega_0\omega^{(2)}a\cos\omega t + \omega^{(2)2}a\cos\omega t$$
$$-\left(1-\frac{\omega_0^2}{\omega^2}\right)\left(-\frac{2}{3}\alpha a^2\frac{\omega^2}{\omega_0^2}\cos(2\omega t)\right) - \left(1-\frac{\omega_0^2}{\cancel{\omega^2}}\right)\ddot{x}^{(3)}$$

$$\ddot{x}^{(3)} + \omega_0^2 x^{(3)} = \frac{\alpha a^2}{2} + \left(+\frac{\alpha^2 a^3}{\omega_0^2} + 2\omega_0\omega^{(2)}a + \omega^{(2)2}a\right)\cos\omega t - \alpha a^2\cos^2\omega t - \beta a^3\cos^3\omega t$$
$$+\left(\frac{\alpha a^2}{2} + \frac{2}{3}\alpha a^2\frac{\omega^2}{\omega_0^2} - \frac{2}{3}\alpha a^2\right)\cos(2\omega t) - \frac{\alpha^2 a^3}{3\omega_0^2}\cos\omega t\cos(2\omega t)$$

terms with $\sim$ neglected because $O(\varepsilon^3)$ or higher, terms with / cancelled with another term

1. (cont'd)

$$\ddot{x}^{(3)} + w_0^2 x^{(3)} = \cancel{\frac{\alpha a^2}{2}} + \left(+ \frac{\alpha^2 a^3}{w_0^2} + 2w_0 w^{(2)} a + \cancel{w^{(1)2} a} \right)\cos wt - \cancel{\frac{\alpha a^2}{2}} - \cancel{\frac{\alpha a^2}{2}\cos(2wt)}$$

$$- \frac{3}{4}\beta a^3 \cos wt - \frac{\beta a^3}{4}\cos(3wt)$$

$$+ \left(\cancel{\frac{\alpha a^2}{2}} + \cancel{\frac{2}{3}\alpha a^2} + \cancel{\frac{1}{3}\alpha a^2 \frac{w^{(1)}}{w_0}} + \cancel{\frac{2}{3}\alpha a^2 \frac{w^{(1)2}}{w_0^2}} - \cancel{\frac{2}{3}\alpha a^2} \right)\cos(2wt)$$

$$- \frac{\alpha^2 a^3}{6w_0^2}\cos wt - \frac{\alpha^2 a^3}{6w_0^2}\cos(3wt)$$

$$= \left(\frac{5}{6}\frac{\alpha^2 a^3}{w_0^2} + 2w_0 w^{(2)} a - \frac{3}{4}\beta a^3 \right)\cos wt + \left(-\frac{\beta a^3}{4} - \frac{\alpha^2 a^3}{6w_0^2} \right)\cos(3wt)$$

→ need the resonant term to go to zero:

$$\frac{5}{6}\frac{\alpha^2 a^3}{w_0^2} + 2w_0 w^{(2)} a - \frac{3}{4}\beta a^3 = 0$$

$$2w_0 w^{(2)} = \frac{3}{4}\beta a^2 - \frac{5}{6}\frac{\alpha^2 a^2}{w_0^2}$$

$$\boxed{w^{(2)} = \frac{3}{8}\frac{\beta a^2}{w_0} - \frac{5}{12}\frac{\alpha^2 a^2}{w_0^3}}$$ ← frequency shift

$$\ddot{x}^{(3)} + w_0^2 x^{(3)} = \left(-\frac{\beta a^3}{4} - \frac{\alpha^2 a^3}{6w_0^2} \right)\cos(3wt)$$

let $x^{(3)} = A + B\cos(3wt)$

$\dot{x}^{(3)} = -3wB\sin(3wt)$

$\ddot{x}^{(3)} = -9w^2 B\cos(3wt)$

$\approx -9w_0^2 B\cos(3wt)$ $\qquad w^2 = (w_0 + w^{(2)}\varepsilon^2) \approx w_0^2$

$$(-9w_0^2 B + w_0^2 B)\cos(3wt) + w_0^2 A = \left(-\frac{\beta a^3}{4} - \frac{\alpha^2 a^3}{6w_0^2} \right)\cos(3wt)$$

$\Rightarrow A = 0$

$$-8w_0^2 B = -\frac{\beta a^3}{4} - \frac{\alpha^2 a^3}{6w_0^2} \Rightarrow B = \frac{\beta a^3}{32w_0^2} + \frac{\alpha^2 a^3}{48w_0^4}$$

$$\boxed{x^{(3)} = \left(\frac{\beta a^3}{32w_0^2} + \frac{\alpha^2 a^3}{48w_0^4} \right)\cos(3wt)}$$

PHYS 200B

HW #2 Problem #2 Kurt Thompson

1/21/15

2) a) $\varepsilon^2 \psi'' + Q(x)\psi = 0$

Use ansatz of the form $\psi \sim \exp\left[\frac{i}{\varepsilon}\sum_{n=0}^{\infty} \varepsilon^n \phi_n(x)\right]$ up to $n=2$

$\psi \sim e^{\frac{i}{\varepsilon}\phi_0} e^{i\phi_1} e^{i\varepsilon\phi_2}$

$\psi' = \frac{i}{\varepsilon}\phi_0' e^{\frac{i}{\varepsilon}\phi_0} e^{i\phi_1} e^{i\varepsilon\phi_2} + e^{\frac{i}{\varepsilon}\phi_0}\{ i\phi_1' e^{i\phi_1} e^{i\varepsilon\phi_2} + i\varepsilon\phi_2' e^{i\phi_1} e^{i\varepsilon\phi_2}\}$

$\psi' = \left(\frac{i}{\varepsilon}\phi_0' + i\phi_1' + i\varepsilon\phi_2'\right)\psi$

$\psi'' = \left(\frac{i}{\varepsilon}\phi_0'' + i\phi_1'' + i\varepsilon\phi_2''\right)\psi + \left(\frac{i}{\varepsilon}\phi_0' + i\phi_1' + i\varepsilon\phi_2'\right)\psi'$

$\psi'' = \left(\frac{i}{\varepsilon}\phi_0'' + i\phi_1'' + i\varepsilon\phi_2''\right)\psi + \left(\frac{i}{\varepsilon}\phi_0' + i\phi_1' + i\varepsilon\phi_2'\right)^2\psi$

$\psi'' = \left(\frac{i}{\varepsilon}\phi_0'' + i\phi_1'' + i\varepsilon\phi_2''\right)\psi - \left[\frac{1}{\varepsilon^2}(\phi_0')^2 + (\phi_1')^2 + \varepsilon^2(\phi_2')^2 + \frac{2}{\varepsilon}\phi_0'\phi_1' + 2\phi_0'\phi_2' + 2\varepsilon\phi_1'\phi_2'\right]\psi$

Substitute into the differential equation and divide by ψ:

$$\varepsilon^2\psi'' + Q\psi = 0$$

$i\varepsilon\phi_0'' + i\varepsilon^2\phi_1'' + i\varepsilon^3\phi_2'' - (\phi_0')^2 - \varepsilon^2(\phi_1')^2 - \varepsilon^4(\phi_2')^2 - 2\varepsilon\phi_0'\phi_1' - 2\varepsilon^2\phi_0'\phi_2' - 2\varepsilon^3\phi_1'\phi_2'$
$+ Q = 0$

Combine terms according to their power of ε:

$(Q - (\phi_0')^2) + (i\phi_0'' - 2\phi_0'\phi_1')\varepsilon + (i\phi_1'' - (\phi_1')^2 - 2\phi_0'\phi_2')\varepsilon^2$
$+ (i\phi_2'' - 2\varepsilon^2\phi_1'\phi_2')\varepsilon^3 - (\phi_2')^2\varepsilon^4 = 0$

Since ε is an arbitrary parameter, each coefficient of a power of ε must independently vanish resulting in the following coupled equations.

(1) $Q - (\phi_0')^2 = 0$

(2) $i\phi_0'' - 2\phi_0'\phi_1' = 0$

(3) $i\phi_1'' - (\phi_1')^2 - 2\phi_0'\phi_2' = 0$

b) Equation (1) is the eikonal equation for ϕ_0.

Equation (2) is called the transport equation.

c) Solve equation (1) first:

$$(\phi_0')^2 = Q$$
$$\phi_0' = \pm\sqrt{Q}$$
$$\phi_0 = \pm\int_{x_0}^{x} dt\,\sqrt{Q(t)} \qquad \text{by Leibniz integral rule.}$$
$$x_0 \in \mathbb{R}$$

In order to solve (2) we need ϕ_0'':

$$\phi_0' = \pm\sqrt{Q}$$
$$\phi_0'' = \pm\frac{1}{2}Q'Q^{-\frac{1}{2}}$$

Solve equation (2):

$$i\phi_0'' - 2\phi_0'\phi_1' = 0$$
$$\pm\frac{i}{2}Q'Q^{-\frac{1}{2}} \mp 2Q^{\frac{1}{2}}\phi_1' = 0$$
$$\mp 2Q^{\frac{1}{2}}\phi_1' = \mp\frac{i}{2}Q'Q^{-\frac{1}{2}}$$
$$\phi_1' = \frac{i}{4}Q'Q^{-1}$$
$$\phi_1 = \frac{i}{4}\ln|Q| + A \quad , \; A \in \mathbb{C}$$

The approximation for ψ using ϕ_0 and ϕ_1 is given by:

$$\psi \approx e^{\frac{i}{\varepsilon}\int_{x_0}^{x} dt\,\sqrt{Q(t)}}\, e^{-\frac{1}{4}\ln|Q| + iA_1} + e^{-\frac{i}{\varepsilon}\int_{x_0}^{x} dt\,\sqrt{Q(t)}}\, e^{-\frac{1}{4}\ln|Q| + iA_2}$$

$$\psi \approx C_1 Q^{-\frac{1}{4}} e^{\frac{i}{\varepsilon}\int_{x_0}^{x} dt\,\sqrt{Q(t)}} + C_2 Q^{-\frac{1}{4}} e^{-\frac{i}{\varepsilon}\int_{x_0}^{x} dt\,\sqrt{Q(t)}} \qquad C_1, C_2 \in \mathbb{C}$$

This approximation is not valid for $Q(x)=0$. (Turning points)
If $Q(x) > 0$ then the solution ψ is an oscillating function.
If $Q(x) < 0$ then the solution ψ has amplitudes that go as exponentials and can therefore possibly diverge.

d) Since the series $\sum_{n=0}^{\infty} \epsilon^{n-1}\phi_n(x)$ is in general a divergent series, the series needs to be truncated properly to produce a good approximation for ψ. Two general conditions need to be met in order for a good approximation:

$$(1) \quad \varepsilon^{n}\phi_{n+1}(x) << \varepsilon^{n-1}\phi_n(x) \quad \text{and} \quad (2) \quad \varepsilon^{n}\phi_{n+1}(x) << 1 \quad \text{as } \varepsilon \to 0$$

The first condition indicates that each term in the series must be much smaller than the previous term. The second condition states that the term after the truncated term must be much smaller than 1. For this problem the conditions become:

$$(1) \quad \varepsilon\,\phi_2(x) << \phi_1(x) << \tfrac{1}{\varepsilon}\phi_0(x) \qquad \varepsilon \to 0$$

$$(2) \quad \varepsilon\,\phi_2(x) << 1 \qquad \varepsilon \to 0$$

3.)

Question: Consider an acoustic wave propagating in a 3D medium with index of refraction $n = n(x, y, z)$

a.)

For a short wavelength, constant frequency excitation, so that the ray theory is applicable, discuss under what circumstances you will be able to solve the ray equations.

We begin with the wave equation:

$$\frac{\partial^2 \psi}{\partial \mathbf{x}^2} + \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0 \tag{1}$$

If we assume an oscillatory solution to Eq (1), and using the relation $n = c_0/c$, then we can write:

$$\begin{aligned} \frac{\partial^2 \psi}{\partial \mathbf{x}^2} + \frac{\omega^2}{c^2}\psi = 0 \\ \frac{\partial^2 \psi}{\partial \mathbf{x}^2} + \frac{\omega^2}{c_0^2}\mathbf{n}^2(x, y, z)\psi = 0 \end{aligned} \tag{2}$$

Now, the problem states that the light we are concerned with has a short wavelength. Thus the wave can be treated as a ray and we can approximate $\psi(\mathbf{x})$ as:

$$\begin{aligned} \psi(\mathbf{x}) &= A(\mathbf{x}) \exp\left[\frac{i}{\epsilon}\sum_{m=0}^{\infty} \epsilon^m \phi_m(\mathbf{x})\right] \\ \psi(\mathbf{x}) &\approx A(\mathbf{x}) \exp\left[\frac{i}{\epsilon}\phi_0(\mathbf{x})\right] \end{aligned} \tag{3}$$

We now plug Eq (3) into the wave equation, Eq (2). I suppress the arguments of the functions for simplicity:

$$\left[-\frac{(\nabla\phi_0)^2}{\epsilon^2}A + \frac{i}{\epsilon}\nabla^2\phi_0 A + \frac{2i}{\epsilon}(\nabla\phi_0 \cdot \nabla A) + \nabla^2 A\right] + \frac{\omega^2}{c_0^2}n^2 A = 0 \tag{4}$$

We now take the $\text{limit}_{\epsilon\to\infty}$. If we also assume that $A(\mathbf{x})$ is slowly varying (so that $\nabla^2 A \approx 0$) the dominate term in Eq (4) becomes:

$$\begin{aligned} -(\nabla\phi_0)^2 A &= -\frac{\omega^2}{c_0^2}n^2 A \\ (\nabla\phi_0)^2 &= \frac{\omega^2}{c_0^2}n^2 \end{aligned} \tag{5}$$

Which is just the Eikonal equation as expected. Expanding the previous equation out:

$$\left(\frac{\partial\psi_0}{\partial x}\right)^2 + \left(\frac{\partial\psi_0}{\partial y}\right)^2 + \left(\frac{\partial\psi_0}{\partial z}\right)^2 = \frac{\omega^2}{c_0^2}n^2(x,y,z) \tag{6}$$

Thus, we will only be able to solve the ray equations if:

$$\boxed{\phi_0(\mathbf{x}) = \phi_x(x) + \phi_y(y) + \phi_z(z)}$$

We furthermore need $n^2(x,y,z)$ to respect this structure as well, or

$$\boxed{n^2(x,y,z) = a(x) + b(y) + c(z)}$$

b.)

Question: In the case of a.), calculate the Eikonal phase $\phi(\mathbf{x})$

Assuming the condition of part a.) are true:

$$\left(\frac{\partial\phi_x}{\partial x}\right)^2 + \left(\frac{\partial\phi_y}{\partial y}\right)^2 + \left(\frac{\partial\phi_z}{\partial z}\right)^2 = \frac{\omega^2}{c_0^2}\left[a(x) + b(y) + c(z)\right]$$

$$\left\{\left(\frac{\partial\phi_x}{\partial x}\right)^2 - \frac{\omega^2}{c_0^2}a(x)\right\} + \left\{\left(\frac{\partial\phi_y}{\partial y}\right)^2 - \frac{\omega^2}{c_0^2}b(y)\right\} + \left\{\left(\frac{\partial\phi_z}{\partial z}\right)^2 - \frac{\omega^2}{c_0^2}c(z)\right\} = 0$$

For the above to be true at all points in space, each of the bracketed terms must be equal to some constant. Let's define:

$$\left\{\left(\frac{\partial\phi_x}{\partial x}\right)^2 - \frac{\omega^2}{c_0^2}a(x)\right\} = f_1$$

$$\left\{\left(\frac{\partial\phi_y}{\partial y}\right)^2 - \frac{\omega^2}{c_0^2}b(y)\right\} = f_2$$

$$\left\{\left(\frac{\partial\phi_z}{\partial z}\right)^2 - \frac{\omega^2}{c_0^2}c(z)\right\} = f_3$$

Where f_i is a constant for $i = 1,2,3$. We can thus write:

$$\frac{\partial\phi_x}{\partial x} = \pm\sqrt{f_1 + \frac{\omega^2}{c_0^2}a(x)}$$

$$\frac{\partial\phi_y}{\partial y} = \pm\sqrt{f_2 + \frac{\omega^2}{c_0^2}b(y)}$$

$$\frac{\partial\phi_z}{\partial z} = \pm\sqrt{-(f_1 + f_2) + \frac{\omega^2}{c_0^2}c(z)}$$

We can now integrate these equations to arrive at the final form for the Eikonal phase, $\phi(\mathbf{x})$:

$$\boxed{\phi(\mathbf{x}) = \pm \int dx \sqrt{f_1 + \frac{\omega^2}{c_0^2} a(x)} \pm \int dy \sqrt{f_2 + \frac{\omega^2}{c_0^2} b(y)} dy \pm \int dz \sqrt{-(f_1 + f_2) + \frac{\omega^2}{c_0^2} c(z)}}$$

HW #2
Problem #4 Solution

{ Determine the variation w/ altitude of a sound wave propagating in isothermal atmosphere with gravity.

- The speed of sound has the temperature dependence (ideal gas)

$C = \sqrt{\frac{\gamma \cdot K \cdot T}{m}}$, so assuming that the composition and temperature of the gas are constant, the speed of the sound wave is constant, call it "c".

- Secondly, by energy conservation, the sound energy density flux has divergence zero, assuming negligible dissipative losses,

$$\boxed{\nabla \cdot \vec{q} = 0.}$$

where the mean sound energy density is:

$u = \rho_0 \bar{v}^2$, so the sound energy flux is:

$\bar{q} = \rho_0 \bar{v}^2 \cdot c \hat{n}$.

using the relation of |Flux| = energy density · speed.

Therefore,

$$\nabla \cdot \vec{q} = \vec{\nabla}(\rho \bar{v}^2 \cdot c \hat{n}). = c(\rho \bar{v}^2 \hat{n})$$

and assuming a spherical wave propagating radially,

$\bar{v}^2 = \bar{v}^2(r)$,

whereby:

$$c\,\vec{\nabla}\,(\rho \bar{v}^2 \hat{n}) = 0$$

$$c\,\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \rho \bar{v}^2\right)\right) = 0$$

so $\rho \bar{v}^2 \sim 1/r^2$. , r is distance from source

- By the barometric formula,

$$\rho \sim e^{-\mu g z / kT}$$

and so,

$$\boxed{1/\bar{v} \sim \sqrt{\rho}\cdot r \sim r e^{-\mu g z / 2kT}}$$

5. Soln:

Using Fermat's minimum time principle

$$T = \int \frac{ds}{v} = \frac{1}{v_0}\int_{x_1}^{x_2} n(\vec{x})\, ds \quad , \quad n = \frac{v_0}{v}$$

$$0 = \delta T = \frac{1}{v_0}\int \left(\frac{\partial n}{\partial \vec{x}}\,\delta\vec{x}\, ds + n\,\delta ds\right)$$

$$(ds)^2 = (d\vec{x})^2 \quad \text{since distance is same}$$

do variation on both sides

$$ds \cdot \delta ds = d\vec{x}\cdot \delta d\vec{x}$$

plug δds into the equation above, and integrate by part,

$$\delta T = \frac{1}{v_0}\, n\,\frac{d\vec{x}}{ds}\cdot\delta\vec{x}\,\Big|_{x_1}^{x_2} + \frac{1}{v_0}\int_{x_1}^{x_2}\left\{\frac{\partial n}{\partial \vec{x}} - \frac{d}{ds}\left(n\frac{d\vec{x}}{ds}\right)\right\} ds\cdot\delta\vec{x}$$

because endpoints x_1, x_2 are fixed, ∴ first term vanishes

$$0 = \delta T = \frac{1}{v_0}\int_{x_1}^{x_2}\left\{\frac{\partial n}{\partial \vec{x}} - \frac{d}{ds}\left(n\frac{d\vec{x}}{ds}\right)\right\} ds\cdot\delta\vec{x}$$

$$\Rightarrow \quad \frac{\partial n}{\partial \vec{x}} - \frac{d}{ds}\left(n\frac{d\vec{x}}{ds}\right) = 0$$

expand the equation

$$\frac{d^2\vec{x}}{ds^2} = \frac{1}{n}\frac{\partial n}{\partial \vec{x}} - \frac{1}{n}\left(\frac{\partial n}{\partial \vec{x}}\cdot\frac{d\vec{x}}{ds}\right)\frac{d\vec{x}}{ds}$$

$$= \frac{1}{n}\nabla n - \frac{1}{n}(\nabla n\cdot\hat{t})\,\hat{t} \qquad \hat{t} \text{ is unit tangent vector}$$

$$= \frac{1}{n}(\nabla n\cdot\hat{n}_0)\,\hat{n}_0 \qquad \hat{n}_0 \text{ is unit normal vector}$$

and $\frac{d^2\vec{x}}{ds^2}$ is curvature

∴ acoustic path curves toward region of increasing index.

For particle, we'll start with Maupertuis ~~Equa~~ principle

$$0 = \delta S_0 = \delta \int p_i dq_i$$

one thing to remember is we choose the path that conserves energy.

$$L = \frac{1}{2} a_{i,k} \dot{q}_i \dot{q}_k - U(\vec{r})$$

$$\therefore \quad P_i = \frac{\partial L}{\partial \dot{q}_i} = a_{i,k} \dot{q}_k$$

$$S_0 = \int a_{i,k} \, dq_k \, dq_i / dt$$

for dt

$$E = \frac{1}{2} a_{i,k} dq_i dq_k / (dt)^2 + U(\vec{r})$$

$$dt = \left(\frac{a_{i,k} \, dq_i \, dq_k}{2(E-U)} \right)^{\frac{1}{2}}$$

plug in

$$S_0 = \int \left(2(E - U(\vec{r})) \cdot a_{i,k} \, dq_i \, dq_k \right)^{\frac{1}{2}}$$

define $a_{i,k} \, dq_i \, dq_k = (dl)^2$

$$\therefore \quad S_0 = \int_{\vec{r}_1}^{\vec{r}_2} \sqrt{2(E - U(\vec{r}))} \, dl$$

which is similar to Fermat's Principle

$$T = \frac{1}{v_0} \int_{\vec{x}_1}^{\vec{x}_2} n(\vec{x}) \, ds$$

$$\sqrt{2(E - U(\vec{r}))} \rightarrow n(\vec{x})$$

$$dl \rightarrow ds$$

$$\vec{r} \rightarrow \vec{x}$$

so we can do the same procedure to particle motion

eventually, we get

$$\frac{d^2 \vec{r}}{dl^2} = \frac{1}{2(E-U)} \left(-\frac{\partial U}{\partial \vec{r}} \cdot \hat{n}_0 \right) \hat{n}_0 \quad , \quad \text{where } -\frac{\partial U}{\partial \vec{r}} = \vec{F}(\vec{r})$$

$\hat{n}_0$ is unit normal vector

$$\therefore \text{ curvature } \frac{d^2 \vec{r}}{dt^2} = \frac{1}{2(E - U(\vec{r}))} \left(\vec{F}(\vec{r}) \cdot \hat{n}_0 \right) \cdot \hat{n}_0$$

6. a). The general condition on the potential $V(r,\phi,z)$ for separability of H-J eqn. in cylindrical coordinates?

b). Solve the H-J eqn. by separation.

a).

$$H = \frac{1}{2m}P_r^2 + \frac{1}{2m}P_z^2 + \frac{1}{2mr^2}P_\phi^2 + V(r,\phi,z) = \frac{1}{2m}\left(\frac{dS}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dS}{d\phi}\right)^2 + \frac{1}{2m}\left(\frac{dS}{dz}\right)^2 + V(r,\phi,z) = E$$

$$S = S_1(r) + S_2(\phi) + S_3(z) \quad , \quad \frac{1}{2m}\left\{\left(\frac{dS_1}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_2}{d\phi}\right)^2 + \left(\frac{dS_3}{dz}\right)^2\right\} + V(r,\phi,z) = E$$

To make H-J eqn. separable, $V(r,\phi,z) = a(r) + \frac{1}{r^2}b(\phi) + c(z)$

$$\therefore \cancel{\frac{1}{2m}\left\{\left(\frac{dS_1}{dr}\right)^2\right.} \quad \underbrace{\left\{\frac{1}{2m}\left(\frac{dS_1}{dr}\right)^2 + a(r)\right\}}_{f_1(r)} + \frac{1}{r^2}\underbrace{\left\{\frac{1}{2m}\left(\frac{dS_2}{d\phi}\right)^2 + b(\phi)\right\}}_{f_2(\phi)} + \underbrace{\left\{\frac{1}{2m}\left(\frac{dS_3}{dz}\right)^2 + c(z)\right\}}_{f_3(z)} = E$$

b).

Let
$$\begin{cases} f_2(\phi) = C_\phi \\ f_1(r) + \frac{C_\phi}{r^2} = C_r \\ f_3(z) + C_r = E \end{cases}$$

Then
$$S_2(\phi) = \int \sqrt{2m}\,(C_\phi - b(\phi))^{1/2} d\phi$$
$$S_1(r) = \sqrt{2m}\int \left(C_r - a(r) - \frac{P_\phi^2}{r^2}\right)^{1/2} dr$$
$$S_3(z) = \sqrt{2m}\int \left(E - c(z) - C_r\right)^{1/2} dz$$

If $b(\phi) = 0$. $S_2(\phi) = P_\phi \phi + \text{const}$. $C_\phi = P_\phi^2/2m$

7. Soln:

a) let $\breve{\rho} = \rho_1 e^{i\phi}$, $\breve{v} = \vec{v}_1 \cdot e^{i\phi}$, where $\phi = \vec{k}\cdot\vec{x} - \omega t$

plug into linearied acoustic equation, we have then

$$\begin{cases} i\rho_1 e^{i\phi}\partial_t\phi + i\rho_1 e^{i\phi}\vec{v}\cdot\nabla\phi = -\rho_0 i e^{i\phi}\vec{v}_1\cdot\nabla\phi \\ \rho_0[\vec{v}_1\cdot ie^{i\phi}\partial_t\phi + ie^{i\phi}\vec{v}_1(\vec{v}\cdot\nabla\phi)] = -C_s^2\cdot ie^{i\phi}\rho_1\nabla\phi \end{cases}$$

multiply these two equations,

$$(\partial_t\phi + \vec{v}\cdot\nabla\phi)^2 = (\nabla\phi)^2 C_s^2(x) \qquad \text{eikonal equation}$$

And $\partial_t\phi = -\omega \quad \nabla\phi = \vec{k}$

$$\therefore \quad (-\omega + \vec{k}\cdot\vec{v})^2 = k^2 C_s^2(x)$$

which means there is a frequency shift.

b) write phase ϕ in flow frame

$$\phi = \int \vec{k}\cdot(d\vec{x} + \vec{v}dt) - \omega t$$
$$= \int \vec{k}\cdot d\vec{x} - (\omega - \vec{k}\cdot\vec{v})dt$$

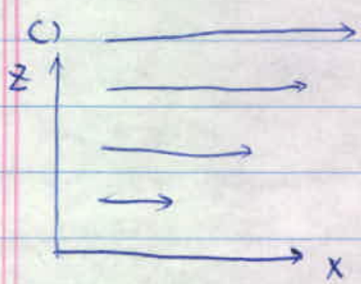
we can consider $\omega - \vec{k}\cdot\vec{v}$ as a new frequency.

use eikonal equation, we have

$$\frac{\partial\,\omega - \vec{k}\cdot\vec{v}}{\partial\vec{k}} = \frac{\partial w}{\partial\vec{k}} - \vec{v} = \frac{d\vec{x}}{dt}$$

$$-\frac{\partial w - \vec{k}\cdot\vec{v}}{\partial\vec{x}} = -\frac{\partial w}{\partial\vec{x}} + \frac{\partial\vec{k}\cdot\vec{v}}{\partial\vec{x}} \cancel{= -\frac{\partial w}{\partial x} + \vec{k}\frac{\partial\vec{v}}{\partial\vec{x}}} = \frac{d\vec{k}}{dt}$$

c)



$$\vec{V} = V(z)\cdot\hat{x}$$

$$\frac{d\vec{k}}{dt} = -\frac{\partial w}{\partial\vec{x}} + \frac{\partial\vec{k}\cdot\vec{v}}{\partial\vec{x}}$$

$$= -\frac{\partial w}{\partial\vec{x}} + \frac{\partial k_x\cdot V(z)}{\partial z}\cdot\hat{z}$$

$$\approx \frac{\partial V(z)}{\partial z}\cdot k_x\cdot\hat{z}$$

it shows that $\vec{k}$ is curved to $\hat{z}$, which means the sound begins to go upwards or downward. So we can't hear that sound at a distance on windy day.

d. ray equations are hamiltonian, is equivalent to phase space flow is incompressible

$$\nabla \cdot \vec{V}_T = 0$$

here we have

$$\frac{\partial}{\partial \vec{k}} \frac{d\vec{k}}{dt} + \frac{\partial}{\partial \vec{x}} \cdot \frac{d\vec{x}}{dt}$$

$$= \frac{\partial}{\partial \vec{k}} \left(-\frac{\partial w}{\partial \vec{x}} + \frac{\partial \vec{k} \cdot \vec{v}}{\partial \vec{x}} \right) + \frac{\partial}{\partial \vec{x}} \left(\frac{\partial w}{\partial \vec{k}} - \vec{v} \right)$$

$$= \frac{\partial}{\partial \vec{x}} \frac{\partial \vec{k} \vec{v}}{\partial \vec{k}} - \frac{\partial^2 w}{\partial \vec{k} \partial \vec{x}} + \frac{\partial^2 w}{\partial \vec{k} \partial \vec{x}} - \frac{\partial \vec{v}}{\partial \vec{x}}$$

$$= \frac{\partial \vec{v}}{\partial \vec{x}} - \frac{\partial \vec{v}}{\partial \vec{x}} = 0$$

$\therefore$ phase space flow is incompressible here, $\Rightarrow$ equations is hamiltonian.

8 Non linear Klein Gordon Eqn.

$$\frac{1}{C_0^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + m^2\phi + \alpha\phi^3 = 0$$

a) Look at propagating solutions on the form $\phi(x,t) = \phi(x-ct)$ and reduce the linear problem

Linear KG: $\frac{1}{C_0^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + m^2\phi = 0$

$$\frac{\partial^2\phi}{\partial t^2} = c^2\phi'' \quad , \quad \frac{\partial^2\phi}{\partial x^2} = \phi''$$

$$\frac{c^2}{C_0^2}\phi'' - \phi'' + m^2\phi = 0$$

$$\boxed{(C^2 - C_0^2)\phi'' + m^2 C_0^2\phi = 0}$$ Simplified PDE to ODE!

b) We use the same assumpt ation about ϕ beeing on the form $\phi(x-ct)$. This reduces the problem t

$$(C^2 - C_0^2)\phi'' + m^2 C_0^2\phi + \alpha\phi^3 = 0$$

This equation has the same form as the Duffing Equation, and therefore we use reductive puertubation theory!

c) $C = C^{(0)} + a^2 C^{(2)}$, $\phi = a\phi_1 + a^3\phi_3$ — Same form of solutions as Duffing's

$$\Rightarrow \left((c^{(0)} + a^2 c^{(2)})^2 - c_0^2\right)\left[a_1 \phi_1'' + a^3 \phi_3''\right] + m^2\left[a \phi_1 + a^3 \phi_3\right]$$

$$- \alpha\left[a \phi_1 + a^3 \phi_3\right]^3 = 0$$

$\underline{O(a)}$. $(c^{(0)2} - c_0^2)\,\phi_1'' + m^2 \phi_1 = 0$

$$\Rightarrow \phi_1 = \cos(k(x-ct)) = \underline{\cos(k\xi)}$$

Insert in to linear eq:

$$-(c^{(0)^2} \; c_0^2)\,\phi_1 k^2 - m^2 \phi_1 = 0 \Rightarrow \underbrace{\underline{c^{(0)^2} = c_0^2 + m^2/k^2}}_{\text{Dispersion!}}$$

$\underline{O(a^3)}$:

$$(c^{(0)^2} - c_0^2)\,\phi_3'' + m^2 \phi_3 = -2c^{(0)} c^{(2)} \cos(k\xi)''$$

$$- \alpha(\cos(k\xi))^3$$

Trig. identities allow us to write $\cos(k\xi)^3$ as $-\frac{1}{4}\cos(3k\xi) + \frac{3}{4}\cos(k\xi)$

$$\Rightarrow (c^{(0)^2} - c_0^2)\,\phi_3'' + m^2 \phi_3 = 2k^2 c^{(0)} c^{(2)} \cos(k\xi) - \frac{3}{4}\alpha\cos(k\xi) + \frac{\alpha}{4}\cos(3k\xi)$$

Now we can choose $c^{(2)}$ such tat we kill secularity:

$$2k^2 c^{(0)} c_2 = \frac{3}{4}\alpha \quad , \quad c_2 = \frac{3\alpha}{8c^{(0)}k^2}$$

$$c = c^{(0)} + a^2 c_2 = c^{(0)}\left(1 + a^2 \frac{c_2}{c_{(0)}}\right)$$

$$\boxed{= \left(c_0^2 + \frac{m^2}{k^2}\right)^{1/2}\left[1 + \frac{3a^2\alpha}{8(c_0^2k^2 + m^2)}\right]}$$

(9) forced Duffing Equation

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x + \beta x^3 = f_{ext}/m \qquad (1)$$

$\begin{cases}\text{resonance}\\ \text{damping}\\ \text{nonlinear}\end{cases}$

a)
$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{f_{ext}}{m} = \frac{f}{m}\cos\omega t \qquad (2)$$

$\Rightarrow$ driven damped SHO

$\omega = \omega_{res} + \varepsilon \qquad \omega_{res} = \omega_0$

expect $x(t) = a\cos(\omega t + \phi)$

$\Rightarrow \dot{x}(t) = -a\omega\sin(\omega t + \phi)$

$\Rightarrow \ddot{x}(t) = -a\omega^2\cos(\omega t + \phi)$

$$\Rightarrow (2) \Rightarrow -a\omega^2\cos(\omega t+\phi) - 2\lambda a\omega\sin(\omega t+\phi) + \omega_0^2 a\cos(\omega t+\phi) = \frac{f}{m}\cos(\omega t)$$

we know $\cos(\omega t+\phi) = \cos\omega t\cos\phi - \sin\omega t\sin\phi$

and $\sin(\omega t+\phi) = \sin\omega t\cos\phi + \cos\omega t\sin\phi$

$$\Rightarrow -a\omega^2(\cos\omega t\cos\phi - \sin\omega t\sin\phi) - 2\lambda a\omega(\sin\omega t\cos\phi + \cos\omega t\sin\phi)$$
$$+\omega_0^2 a(\cos\omega t\cos\phi - \sin\omega t\sin\phi) = \frac{f}{m}\cos(\omega t)$$

$$\Rightarrow \cos\omega t\left(-a\omega^2\cos\phi - 2\lambda a\omega\sin\phi + \omega_0^2 a\cos\phi - \frac{f}{m}\right)$$
$$+\sin\omega t\left(+a\omega^2\sin\phi - 2\lambda a\omega\cos\phi - \omega_0^2 a\sin\phi\right) = 0$$

sin and cos are independent so coefficients must vanish

$$\Rightarrow \begin{cases} -a\omega^2\cos\phi - 2\lambda a\omega\sin\phi + \omega_0^2 a\cos\phi - \frac{f}{m} = 0 \\ a\omega^2\sin\phi - 2\lambda a\omega\cos\phi - \omega_0^2 a\sin\phi = 0\end{cases}$$

$$\Rightarrow a = \frac{f/m}{\left((\omega_0^2-\omega^2)^2 + 4\lambda^2\omega^2\right)^{1/2}} = \frac{f}{2m\omega_0(\varepsilon^2+\lambda^2)^{1/2}}$$

dropping terms of 3rd order in ε and λ

b) $\ddot{x} + 2\lambda \dot{x} + \omega_0^2 x + \beta x^3 = \frac{f}{m} \cos \omega t$

for $\beta \neq 0$, $\Rightarrow$ frequency shift from nonlinearity

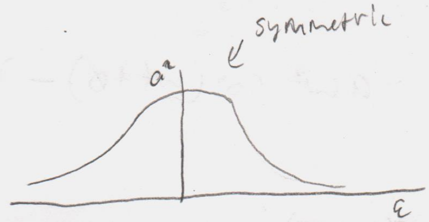
$\Rightarrow \omega_0 \to \omega_0 + \frac{3}{8}\frac{\beta}{\omega_0} a^2 \quad \Rightarrow \omega = \omega_{res} + \varepsilon = \omega_0 + \frac{3}{8}\frac{\beta}{\omega_0} a^2 + \varepsilon$

$$\Rightarrow a = \frac{f/m}{\left(\left(\omega_0 + \frac{3\beta a^2}{8\omega_0}\right)^2 - \omega^2\right)^2 + 4\lambda^2\omega^2}$$

$$\approx \frac{f/m}{2m\omega_0\left(\left(\underbrace{\omega - \omega_0}_{\varepsilon} - \frac{3\beta}{8\omega_0}a^2\right)^2 + \lambda^2\right)^{1/2}} = \frac{f/m}{2m\omega_0\left(\left(\varepsilon - \frac{3\beta}{8\omega_0}a^2\right)^2 + \lambda^2\right)^{1/2}}$$

c) $\beta = 0$

$\Rightarrow a^2 = \frac{f^2}{4m^2\omega_0^2} \frac{1}{\varepsilon^2 + \lambda^2}$



$\beta \neq 0$

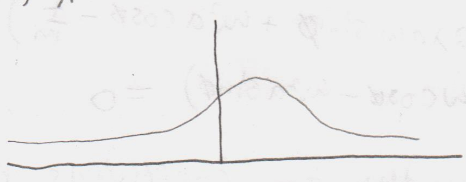
$K = \frac{3\beta}{8\omega_0}$

$$\Rightarrow a^2 = \frac{f^2}{4m^2\omega_0^2\left((\varepsilon - Ka^2)^2 + \lambda^2\right)^{1/2}}$$

$$\Rightarrow \frac{f^2}{4m^2\omega_0^2} = a^2\left((\varepsilon - Ka^2)^2 + \lambda^2\right)^{1/2}$$

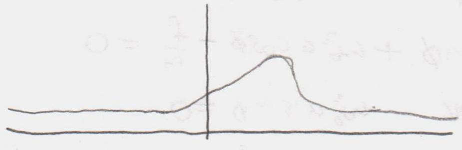
for graphs I set $K = 1$, $\lambda = 2$

for $F = \frac{f^2}{4m^2\omega_0^2} = 1$

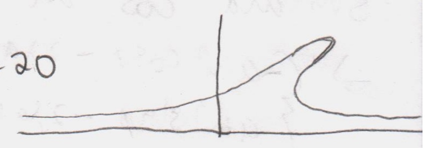


F=15

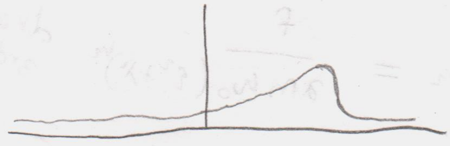
F=5



F=20



F=10



⑨ (continued)

d) $a^2(\varepsilon^2 - 2K\varepsilon a^2 + K^2a^4 + \lambda^2) = \dfrac{f^2}{4m^2\omega_0^2}$

$\Rightarrow$ double-valued when $\dfrac{d\,a^2}{d\varepsilon} = \infty$

Let $F(a^2, \varepsilon) = a^2(K^2a^4 - 2K\varepsilon a^2 + \varepsilon^2 + \lambda^2) - \dfrac{f^2}{4m^2\omega_0^2} = 0$

$\Rightarrow dF = \left(\dfrac{\partial F}{\partial a^2}\right) da^2 + \left(\dfrac{\partial F}{\partial \varepsilon}\right) d\varepsilon = 0$

$\Rightarrow \dfrac{da^2}{d\varepsilon} = \dfrac{-(\partial F/\partial \varepsilon)}{(\partial F/\partial a^2)}$

$\Rightarrow \dfrac{da^2}{da} \to \infty$ when $\dfrac{\partial F}{\partial a^2} = 0$

$\dfrac{dF}{da^2} = \dfrac{d}{da^2}\left\{K^2(a^2)^3 - 2K\varepsilon(a^2)^2 + (\varepsilon^2+\lambda^2)a^2\right\} = 3K^2(a^2)^2 - 4K\varepsilon a^2 + (\varepsilon^2+\lambda^2) = 0$

pythagorean thm w/ Ka^2

$\Rightarrow Ka^2 = \dfrac{4\varepsilon \pm \sqrt{16\varepsilon^2 - 4(3)(\varepsilon^2+\lambda^2)}}{2\cdot 3} = \dfrac{2}{3}\varepsilon \pm \dfrac{1}{3}\sqrt{4\varepsilon^2 - 3\varepsilon^2 - 3\lambda^2} = \dfrac{2}{3}\varepsilon \pm \dfrac{1}{3}\sqrt{\varepsilon^2 - 3\lambda^2}$

$\Rightarrow$ multiple real roots when $\boxed{\varepsilon^2 = 3\lambda^2} \Rightarrow Ka^2 = \dfrac{2\varepsilon}{3}$ this means there is a bifurcation in the path the solution takes

e) $\Rightarrow f^2 = 4m^2\omega_0^2 a^2(\varepsilon^2 - 2\varepsilon K a^2 + K^2a^4 + \lambda^2)$

plugging in $\varepsilon = \sqrt{3}\lambda$ and $Ka^2 = \dfrac{2\varepsilon}{3} = \dfrac{2\sqrt{3}}{3}\lambda$

$\Rightarrow f^2_{crit} = 4m^2\omega_0^2 \dfrac{2\sqrt{3}}{3}\lambda \dfrac{1}{K}\left(3\lambda^2 - 2\sqrt{3}\lambda \dfrac{2\sqrt{3}}{3}\lambda + \dfrac{12}{9}\lambda^2 + \lambda^2\right)$

$= \dfrac{8}{\sqrt{3}} m^2\omega_0^2 \dfrac{\lambda}{K}\left(3\lambda^2 - 4\lambda^2 + \dfrac{4}{3}\lambda^2 + \lambda^2\right) = \boxed{\dfrac{32}{3\sqrt{3}} \dfrac{m^2\omega_0^2\lambda^3}{K}}$